VIBRATION OF CONTINUOUS SYSTEMS

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INTRODUCTION

Modeling the structures with discrete coordinates provides a practical approach for the analysis of structures subjected to dynamic loads. However, the results obtained from these discrete models can only give approximate solutions to the actual behaviour. The present lectures consider the dynamic theory of beams and rods having distributed mass and elasticity for which governing equations of motion are partial differential equations. The integration of these equations is in general more complicated than the solution of ordinary differential equations governing discrete dynamic systems. Due to this mathematical complexity, the dynamic analysis of structures as continuous systems has limited use in practice. Nevertheless, the analysis as continuous systems of some generic models of structures provides very useful information of the overall dynamic behaviour of structures. The method of analysis of continuous system is illustrated with examples of torsional, axial and bending vibration of beams. For, beams with non uniform geometry, closed form solution is very much involved. In such case, approximate methods (such as Rayleigh Ritz method or Gallerkin Method) can be used.

EQUATION OF MOTIONS

The of equations of motion can be derived by (i) force balance of a differential element (ii) Hamilton's principle (iii) Lagranges method. While the force balance is a convenient approach for most of the problems, Hamiltons principle and Lagranges method are applied for a complex system. These two approaches need the consideration of the energy of the system.

Hamilton's Principle

Hamilton's principle is stated as an integral equation in which the energy is integrated over an interval of time. Mathematically, the principle can be stated as

$$\delta\int_{t_1}^{t_2}(T-V)dt=\delta\int_{t_1}^{t_2}Ldt=0 \qquad (1)$$

where L is the Lagrangian, T and V are kinetic and potential energy of the system. The physical interpretation of the eq.(1) is that out of all possible paths of motion of a system during an interval of time from t_1 to t_2 , the actual path will be that for which the integral

$\int_t Ldt$ has a stationary value. It can be shown that in fact stationary value will, in fact, the minimum value of the integral.

The Hamilton's principle can yield the governing differential equations as well as boundary conditions.

Lagrange's Equation

Hamilton's principle is stated as an integral equation where total energy is integrated over an time interval. On the other hand, Lagrange's equation is differential equations, in which one considers the energies of the system instantaneously in time. Hamilton's principle can be used to derive the Lagrange's equation in a set of generalized coordinates. Lagrange's equation may be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right)-\frac{\partial L}{\partial q_i}=0 \qquad (2)$$

TORSIONAL VIBRATION OF BARS

Fig 1(a) represents a non-uniform bar such that the x-axis coincides with the neutral axis where there is no strain. Fig 1(b) shows a free-body diagram for an element dx of the bar.

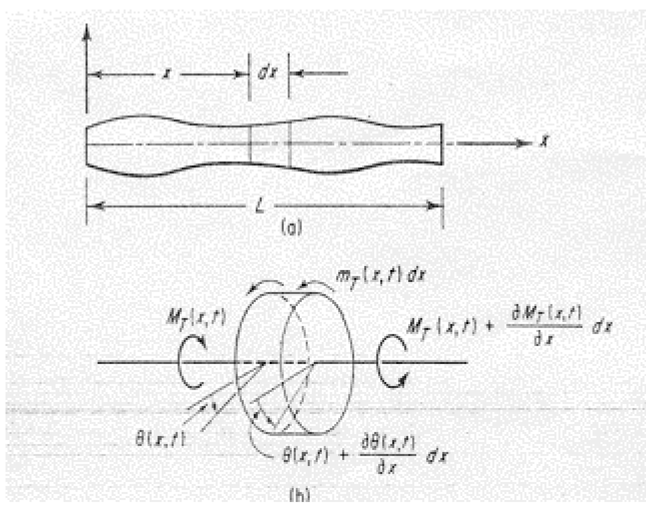

Fig 1 Torsional vibration of shaft

In general, if the cross section is not circular, there is some warping of the cross sectional plane associated with torsional motion. Further more, if the cross-sectional area is circular but the bar is non-uniform, although there is no warping of the cross-sectional planes, the displacements are not proportional to the radial distance from the axis of twist. We shall assume here that the cross-sectional area is uniform such that the motion can be regarded as rotation of the cross sectional planes as a whole and without warping. Let *G* be the shear modulus and let *J(x)* be a geometric property of the cross section, which in the case of a circular cross section at the point x and at time t, the relation between the deformation and the twisting moment is

$$M_T(x,t) = GJ(x)\frac{\partial \theta(x,t)}{\partial x} \tag{3}$$

Where the product GJ (x) is called torsional stiffness.
Let I (x) be the moss polar moment of inertia per unit length of bar $m_T\ (x,t)$ be the external twisting moment per unit length of bar.

The rotational motion of the bar element in the form

$$\left[M_T(x,t) + \frac{\partial M_T(x,t)}{\partial x}dx\right] + m_T(x,t)dx - M_T(x,t) = I(x)dx\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{4}$$

which reduces to

$$\frac{\partial M_T(x,t)}{\partial x} + m_T(x,t) = I(x)dx\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{5}$$

In view of equation (3), we can write equation (5) as

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta(x,t)}{\partial x}\right] + m_T(x,t) = I(x)dx\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{6}$$

which is equation of motion in torsion.
In the case in which $m_T\ (x,t) = 0$, (6) reduces to the equation for the free torsional vibration of a bar,

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta(x,t)}{\partial x}\right] = I(x)dx\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{7}$$

For a clamped end at x = 0, we obtain the boundary condition
θ(0,t)= 0,

(8)

and for a free end at x = L the boundary condition is

$$GJ(x)\frac{\partial \theta(x,t)}{\partial x}\bigg|_{x=L} = 0, \tag{9}$$

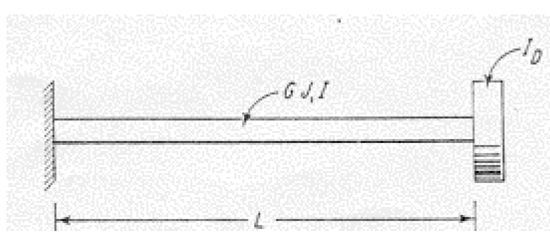

Fig 2 A shaft carrying a rigid disc at free end

Example: Calculation of Natural Frequency of Torsional vibration

Consider a circular bar with a rigid disk attached at one end. The torsional rigidity of the bar is *GJ(x)*, its mass polar moment of inertia per unit length is *I(x)*, and the mass polar moment of the disk is I_D **(Fig 2).**

Boundary conditions, x = 0 we have

$\theta(x,t) = 0$ at x=0

$$GJ(x)\frac{\partial\theta(x,t)}{\partial x}\bigg|_{x=L} = -I_D\frac{\partial^2\theta(x,t)}{\partial^2 t}\bigg|_{x=L} \tag{10}$$

Let $\theta(x,t) = \phi(x)f(t)$ (11)

Recalling that f(t) is harmonic, the igenvalue problem reduces to the differential equation

$$-\frac{d}{dx}\left[GJ(x)\frac{d\phi(x)}{dx}\right] = \omega^2 I(x)\phi(x) \tag{12}$$

to be satisfied throughout the domain 0< x <L, and the boundary conditions

$\phi(0) = 0,$ (13)

$$GJ(x)\frac{\partial\phi(x,t)}{\partial x}\bigg|_{x=L} = \omega^2 I_D\phi(x) \tag{14}$$

at the ends.

For uniform shaft, we have

$$\frac{\omega^2 I}{GJ} = \beta^2 \tag{15}$$

So that the equation (12) reduces to

$$\frac{d^2\phi(x)}{dx^2} + \beta^2\phi(x) = 0, \tag{16}$$

which has the solution

$$\phi(x) = C_1\cos\beta x + C_2\sin\beta x \tag{17}$$

From boundary condition eq.(13), it follows that $C_2 = 0$. Boundary condition (14) gives Thus, the frequency equation is obtained as

$$GJC_1\beta\cos\beta L = \omega^2 I_D C_1\sin\beta L$$

$$\tan\beta L = \frac{IL}{I_D}\frac{1}{\beta L} \tag{18}$$

This is a transcendental equation in βL. It has infinite number of solutions βL which must be obtained numerically and are related to the natural frequencies ω_t by

$$\omega_r = \beta_r L\sqrt{\frac{GJ}{IL^2}}, \qquad r = 1,2\ldots \tag{19}$$

Note that $\omega_r = \beta_r L\sqrt{\frac{GJ}{IL^2}}$, the natural frequencies ω_r are no longer integral multiples of the fundamental frequency ω_1
Natural modes are given by

$$\phi_r(x) = A_r \sin \beta_r x, \qquad r,s = 1,2,\ldots\ldots;\ r\neq s \tag{20}$$

They are orthogonal functions. The orthognality condition follows

$$\int_0^L I\phi_r(x)\phi_s(x)dx + I_D\phi_r(L)\phi_s(L) = 0, \qquad r = 1,2,\ldots\ldots \tag{21}$$

The first three modes or a ratio $I/I_D = 1$ are plotted in Fig 3.

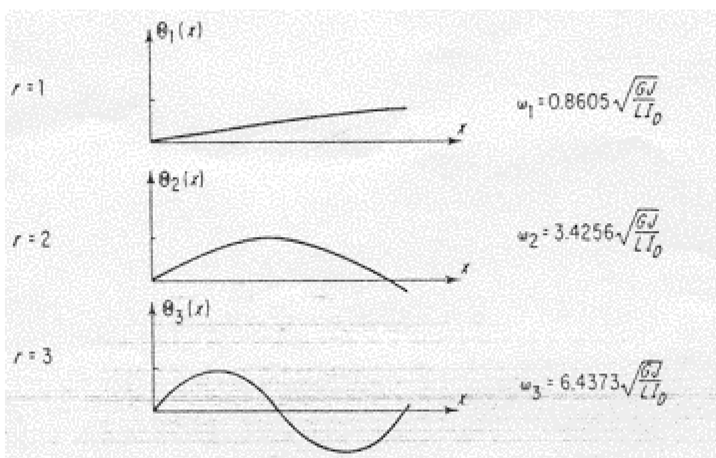

Fig 3 Torsional modes of a cantilever shaft carrying a rigid disk at tip

LONGITUDINAL VIBRATION OF BARS

If *u (x, t)* is the axial displacement and f(x,t) is the imposed axial force, the differential equation of motion is

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] = m(x)\frac{\partial^2 u(x,t)}{\partial t^2} \tag{22}$$

The equation must be satisfied over the domain 0 < x < L. In addition, u must be such that at the end points we have

$$\left(EA\frac{\partial u}{\partial x}\right)\partial u\Big|_0^l = 0 \tag{23}$$

If the bar is clamped at the end x=0, the boundary condition is

$$u(0,t) = 0, \tag{24}$$

and if the end x = l is free, we have

$$EA(x)\frac{\partial u(x,t)}{\partial x}\bigg|_{x=l} = 0. \tag{25}$$

The nature of the differential equation of motion is similar to that of torsional vibration of circular shaft ignoring warping.

Example: Calculation of Natural Frequencies of Axial Vibration

Let us consider a clamped-free rod, for which the eigen-value problem reduces to the solution of the differential equation

Let $u(x,t) = U(x)f(t)$ where U(x) is the mode shape and f(t) is a harmonic function.

$$-\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right] = \varpi^2 m(x)U(x), \tag{26}$$

This must be satisfied throughout the domain, and the homogeneous boundary conditions

$$U(0) = 0, \tag{27}$$

$$EA(x)\frac{dU(x)}{dx}\bigg|_{x=L} = 0, \tag{28}$$

to be satisfied at the end point.

For a uniform rod the eigenvalue problem reduces to the solution of the differential equation

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0, \qquad \beta^2 = \omega^2 \frac{m}{EA}, \tag{29}$$

which is subject to boundary conditions (27) and (28). The solution of (29) is

$$U(x) = C_1 \sin \beta x + C_2 \cos \beta x. \tag{30}$$

Boundary conditions (27) yields $C_2 = 0$, and from boundary conditions (28) we obtain the frequency equation

$$\cos \beta L = 0\,, \tag{31}$$

Which yields the eigenvalues

$$\beta_r = (2r-1)\frac{\pi}{2L}, \quad r = 1,2,.., \tag{32}$$

so that the natural frequencies ϖ_r are

$$\varpi_r = \beta_r \sqrt{\frac{EA}{m}} = (2r-1)\frac{\pi}{2}\sqrt{\frac{EA}{mL^2}}, \quad r = 1,2,.., \tag{33}$$

The corresponding eigen functions have the form

$$U_r(x) = A_r \sin(2r-1)\frac{\pi}{2}\frac{x}{L}, \qquad r = 1,2,.., \tag{34}$$

and they are orthogonal. Let us normalize them and adjust the coefficients A_r such that

$$\int_L^0 m U_r(x) U_s(x)dx = 1, \quad \text{for r=s and 0 for r≠s} \tag{35}$$

From which we obtain the orthonormal set

$$U_r(x) = \sqrt{\frac{2}{mL}} \sin(2r-1)\frac{\pi}{2}\frac{x}{L}, \qquad r = 1,2,.., \tag{36}$$

The first three modes are shown in Figure 4

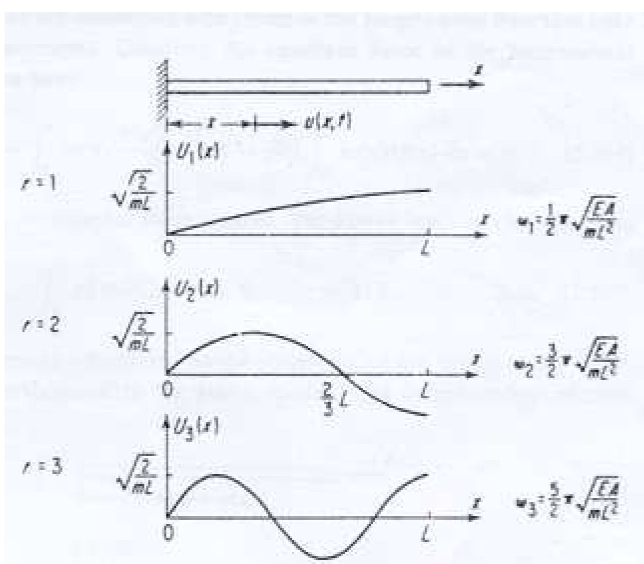

Fig 4 Mode shapes in Axial vibration of Clamped-Free bar

Let us consider a case in which *both ends are free.* The formulations follows the same pattern and once again we find that homogeneous equation must be satisfied through-out the domain 0 < x < L, but in contrast to the previous case the boundary conditions in this case are

$$EA(x)\frac{dU(x)}{dx}\bigg|_{x=0} = 0 \tag{37}$$
$$EA(x)\frac{dU(x)}{dx}\bigg|_{x=L} = 0$$

This problem is self-adjoint and semi definite. Letting the rod be uniform, yield $C_1 = 0$, whereas boundary conditions in (37) gives the frequency equation

$$\sin \beta L = 0, \tag{38}$$

Which leads to the eigen values

$$\beta_r = r\frac{\pi}{l}, \quad r = 0,1,2,\ldots, \tag{39}$$

Here $\beta_o = 0$ is also an eigenvalue, incorporating an earlier statement made in connection with semi definite systems. Corresponding to the eigenvalues other than β_o we have the eigenfunctions

$$U_r(x) = A_r \cos r\pi\frac{x}{L}, \qquad r = 1,2,\ldots. \tag{40}$$

For $\beta = \beta_o = 0$, the equation becomes

$$\frac{d^2U(x)}{dx^2} = 0, \tag{41}$$

which has the solution

$$U_o(x) = A_o + A_o^{'}x. \tag{42}$$

Upon consideration of the boundary conditions, it reduces to

$$U_o(x) = A_o \tag{43}$$

Hence to the eigen values $\beta = \beta_o = 0$ there corresponds a mode that is interpreted as the displacement of the rod as a whole. This is known as a rigid-body mode and is typical of under restrained systems (semi definite systems) for which there are no forces or moment exerted by the supports. In this particular case we are concerned with forces in the longitudinal direction only and not with moments. Denoting the resultant force in the longitudinal direction *F (t)* we have

$$F(t) = \int_o^L m(x)\frac{\partial^2 U(x,t)}{\partial t^2}dx = \ddot{f}(t)\int_0^L m(x)U(x)dx = 0 \tag{44}$$

Because there is no external force present. The above leads to the equations

$$\int_0^L m(x)U_r(x)dx = 0 \qquad r = 1,2,\ldots \tag{45}$$

Which can be merely interpreted as the statement of the fact that the rigid body mode is orthogonal to the elastic modes .The orthogonality relation and the normalization statement may be combined in to

$$\int_0^L m(x)U_r(x)U_s(x)dx = \delta_{rs} \qquad r, s = 0,1,2\ldots \tag{46}$$

The first three modes are plotted in Fig 5. In this case the nodes of $U_r(x)$ are at the points $x = (L/2r)(2n-1)(n = 1,2,....,r)$.

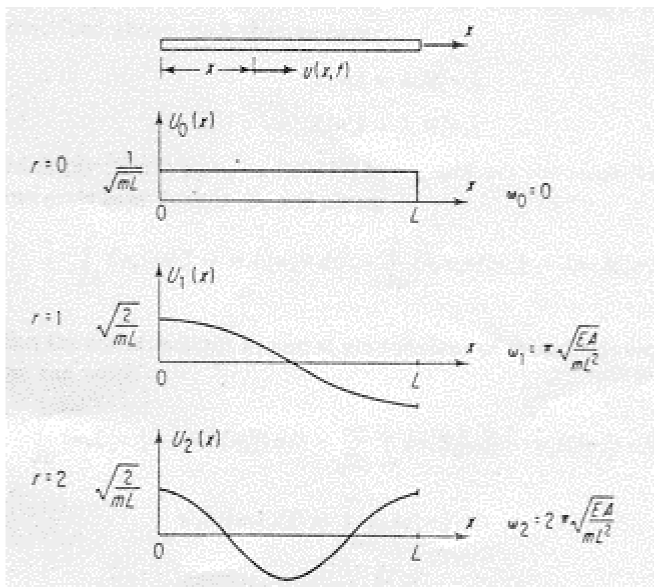

Fig.5 Mode shapes of free-free bar in axial vibration

BENDING VIBRATION OF EULER BERNOULI BEAM

We consider the bar in flexure shown in Fig.6. The transverse displacement at any point x and time t is denoted by y(x,t) and transverse force per unit length by f(x,t). The system parameters are the mass per unit length m(x) and flexural rigidity EI(x). Fig 2 shows the free body diagram corresponding to bar element of length dx. We use so called Euler Bernouli beam theory according to which the rotation of the element is insignificant. This theory is valid if the ratio between the length of the bar and its height is relatively large (say more than 10). In the area of vibration the above statement simply ignores the rotatory inertia and shear deformation effects.
Consideration of the equilibrium of the forces and moments yields the following governing differential equation of motion

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2 y}{\partial x^2}\right\}+m(x)\frac{\partial^2 y}{\partial t^2}=f(x,t) \qquad (47)$$

With inclusion of viscous damping per unit length c the above eq (1) can be modified as

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2 y}{\partial x^2}\right\}+m(x)\frac{\partial^2 y}{\partial t^2}+c(x)\frac{\partial y}{\partial t}=f(x,t) \qquad (48)$$

An elegant technique known as "Mode superimposition" technique exists for the continuous system with linear behaviour. The technique will be discussed in detail with an example. To apply the mode superimposition technique, it is necessary first to know the natural frequencies and corresponding mode shapes

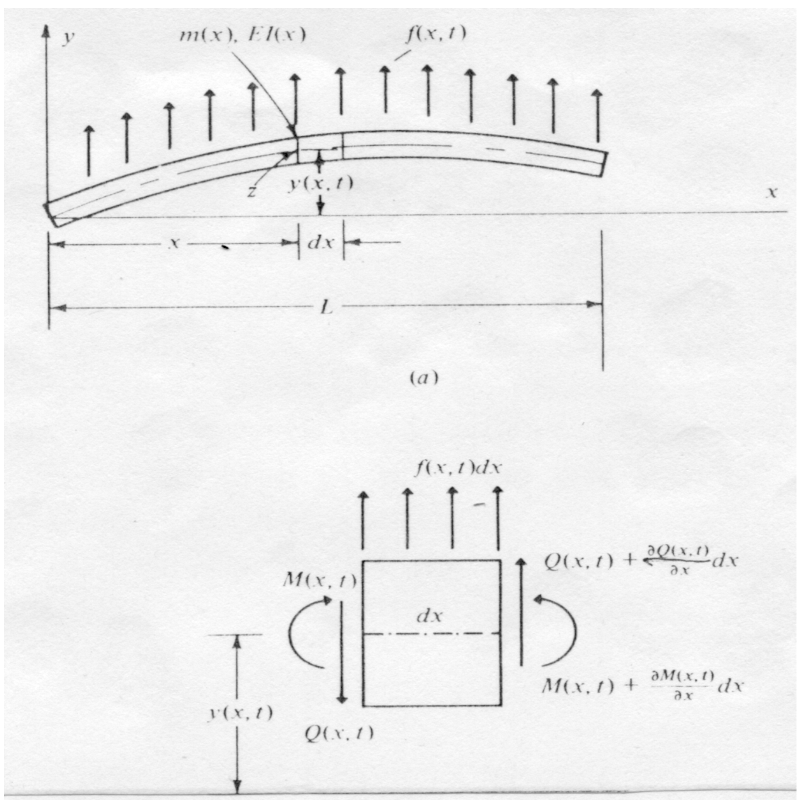

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Fig.6 Bending of beam

NATURAL FREQUENCIES AND MODE SHAPES

The natural frequencies and mode shapes are obtained considering the homogeneous solution of the beam vibration equation. We consider the undamped mode in bending vibration of the beam with uniform sectional property. For free vibration let f(x,t)=0 and assume that the response is given by

$$y(x,t)=\phi(x)\sin\omega t \qquad (49)$$

in which $\phi(x)$ is the mode shape function and ω is the circular natural frequency. Substituting (49) in eq.(47), one has

$$\frac{d^4\phi}{dx^4} - a^4\phi(x) = 0 \qquad (50)$$

The general solution of the equation (50) is

$$\phi(x) = A\sin ax + B\cos ax + C\sinh ax + D\cosh ax \qquad (51)$$

where A, B, C and D are integration constants to be evaluated from the boundary conditions.

Let us consider the beam with *both ends simply supported*. The boundary conditions at the ends imply the following conditions on mode shape functions

$$\phi(0) = \phi(L) = 0$$

$$\phi''(0) = \phi''(L) = 0$$

Substitution of boundary conditions in eq (50) results in following transcendental equation.

$$A\sin aL = 0 \qquad (52)$$

Excluding the trivial solution (A=0), we obtain the frequency equation

sin aL=0 (53)

which will be satisfied for

$a_n L = n\pi$, n=1,2,…

Thus natural frequency in nth mode is

$$\omega_n = (n\pi)^2\sqrt{\frac{EI}{mL^4}} \qquad (54)$$

The mode shape function is

$$\phi_n(x) = \sin\frac{n\pi}{L}x \qquad (55)$$

The procedure is same for the beam with other boundary conditions. The characteristic equations for fixed-fixed beam, clamped-free beam, free-free beam and pinned-clamped beam are given below:

<u>Fixed-fixed beam</u>

$$\cos aL\cosh aL = 1 \qquad (55.a)$$

<u>Clamped-free beam</u>

$$\cos aL\cosh aL + 1 = 0 \qquad (55.b)$$

<u>Free-free beam</u>

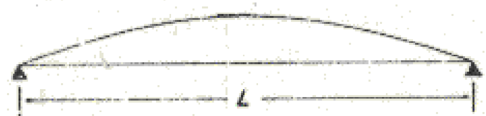
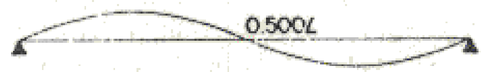
$$\cos aL\cosh aL - 1 = 0 \qquad (55.c)$$

<u>Pinned-clamped</u>

$$\tan aL - \tanh aL = 0 \qquad (55.d)$$

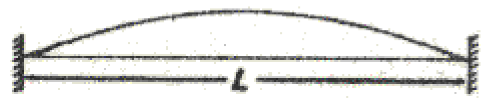
The natural frequencies and mode shapes of simply supported beam, fixed beam and cantilever beams are shown in table 1, 2 and 3 respectively.

Table1 Natural Frequencies and Mode shapes of Simply supported Beam

Natural Frequencies $\omega_n = C_n\sqrt{\dfrac{EI}{mL^4}}$			Normal Modes $\Phi_n = \sin\dfrac{n\pi x}{L}$
n	C_n	I_n^*	Shapes
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0	
5	$25\pi^2$	$4/5\pi$	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi^2(x)\,dx.$

Table2 Natural Frequencies and Mode shapes of Fixed-Fixed Beam

Natural Frequencies		Normal Modes		
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x)$ $\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.000000	0.2323	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi_n^2(x)\,dx.$

Table3 Natural Frequencies and Mode shapes of Fixed-Free Beam

Natural Frequencies

$$\omega_n = C_n\sqrt{\frac{EI}{mL^4}}$$

Normal Modes

$$\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x)$$

$$\sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	3.5160	0.734096	0.7830	
2	22.0345	1.018466	0.4340	
3	61.6972	0.999225	0.2589	
4	120.0902	1.000033	0.0017	
5	199.8600	1.000000	0.0707	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi_n^2(x)\,dx$.

Orthogonality condition

The most important properties of the normal modes is that of orthogonality. It is this property which makes possible the uncoupling of the equations of motion. For two different frequencies $\omega_n \neq \omega_m$, the normal modes must satisfy

$$\int_L m(x)\phi_n(x)\phi_m(x)dx = 0 \text{ for } n \neq m \quad (56)$$

For n=m

$$\int_L m(x)\phi_n^2(x)dx = M_n \quad (57)$$

in which M_n is called the “generalized mass” in the nth mode.

FORCED VIBRATION

Having obtained natural frequencies and mode shapes, the transverse displacement of the beam can be written as

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t) \quad (58)$$

in which η is the generalized coordinates. Theoretically, infinite number of modes for the continuous systems are possible. However, contribution of higher modes towards the response is negligible. Hence in computation only first few modes are considered. Substituting eq.(12) in eq.(2), multiplying both sides by ϕ_k , integrating in the domain of the beam and applying orthogonality condition of normal modes, the partial differential equation of motion can be discretized into uncoupled ordinary differential equation of motion as

$$\ddot{\eta}_i(t) + 2\xi_i\,\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = Q_i \quad \text{(I=1,2,…)} \quad (59)$$

where Qi is the generalized force whose expression is given by

$$Q_i = \frac{1}{M_i}\int_L f(x,t)\phi_i(x)dx \quad (60)$$

Example: Consider a simply supported beam (uniform cross section) subjected to a constant force suddenly applied to a section x_1 from left support.

Generalized force at $x=x_1$

$$Q_i = \frac{1}{M_i}\int_0^L P_0\delta(x-x_1)\phi_i(x)dx \quad (61)$$

where

$$M_i = \int_0^L m\sin^2(\frac{i\pi x}{L})dx = \frac{mL}{2} \quad (62)$$

Thus $Q_i = \dfrac{2}{mL}P_0\sin\dfrac{n\pi x_1}{L}$

For the initial condition of zero displacement and zero velocity, the response of generalized coordinates becomes

$$\eta_i(t) = \frac{2P_0 \sin \frac{n\pi x_1}{L}}{\omega_i^2 mL}(1-\cos \omega_n t)$$

Now using mode superposition technique, the deflection at x at instant t is given by

$$y(x,t) = \frac{2P_0}{mL}\sum_n \frac{1}{\omega^2{}_n} \sin\frac{n\pi}{2}(1-\cos\omega_n t)\sin\frac{n\pi x}{L} \tag{63}$$

It is apparent that all even the modes do not contribute to the deflection. It is also of interest to compare the contribution of various modes. This comparison can be done on the basis of maximum modal displacement disregarding the manner in which these displacements combine. The amplitude will indicate the relative importance of the modes. The dynamic load factor ($1-\cos\omega_n t$) has a maximum value of 2 for all the modes. Furthermore, for all modes (except even modes), the modal contribution is simply in proportion to $1/\omega^2{}_n$. Therefore in higher modes the factor $1/\omega^2{}_n$ becomes small and hence its contribution can be ignored in the superimposition of modes.

MULTISPAN BEAM

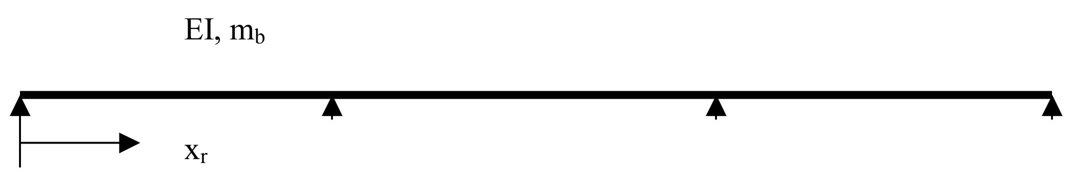


Fig.7 Multispan beam

Consider the multispan beam as shown in Fig. 7. With uniform EI , m, c, the equation of transverse vibration for each span is given by

$$EI\frac{\partial^4 y_r}{\partial x_r{}^4} + c\frac{\partial y_r}{\partial t} + m_b\frac{\partial^2 y_r}{\partial t^2} = f_r(x_r, t) \tag{64}$$

$$0 \le x_r \le l_r \qquad r = 1,2,3,......N.$$

In which the suffix r denotes the r^{th} span; EI, m_b and c denotes the flexural rigidity, mass and viscous damping per unit length respectively. Furthermore, y_r is the transverse deflection on r^{th} span, $f(x_r,t)$ is the time-varying external load distribution due to moving loads, x_r is the local co-ordinate along the axis of the r^{th} span at instant *t*.

The homogeneous solution of the Eqn. (64) ignoring damping is given by

$$\phi_{nr}(x) = A_{nr}\sin\beta_{nr}x_r + B_{nr}\cos\beta_{nr}x_r + C_{nr}\sinh\beta_{nr}x_r + D_{nr}\cosh\beta_{nr}x_r \tag{65}$$

Where A_{nr}, B_{nr}, C_{nr} and D_{nr} are the integration constants, $\phi_{nr}(x)$ is eigen function of the n^{th} mode of the r^{th} span. The frequency parameter β_{nr} in the n^{th} mode is given by

$$\beta_{nr}{}^4 = \frac{m\omega_{nr}{}^2}{EI} \tag{66}$$

in which ω_{nr} is the natural frequency of the beam (rad/sec). The following boundary conditions of continuous beam need to be applied:

$$y_r(x_r = l_r, t) = 0 \tag{67}$$

$$y_{(r+1)}(x_{r+1} = l_r, t) = 0 \tag{68}$$

$$\frac{\partial y_r}{\partial x_r}(x_r = l_r, t) = \frac{\partial y_{(r+1)}}{\partial x_{(r+1)}}(x_{r+1} = 0, t) \tag{69}$$

$$\frac{\partial^2 y_r}{\partial x^2{}_r}(x_r = l_r, t) = \frac{\partial^2 y_{(r+1)}}{\partial x^2{}_{(r+1)}}(x_{r+1} = 0, t) \qquad r = 1,2,3...N \tag{70}$$

Using the boundary conditions in Eqn (65), a set of homogeneous equations can be found in the matrix form as

$$[V(\beta_{nr})]\{W\} = \{0\} \tag{71}$$

The non-trivial solution of the Eqn. (71) necessitates that the determinant of the matrix $[V(\beta_{nr})]$ should be equal to zero. After expanding the determinant the characteristic polynomial can be solved to find the frequency roots which when substituted in Eqn. (71) yields the vector {W} and hence the mode shape.

The mode shape function for multi-span continuous beam is given by

$$\phi_{nr}(x) = \begin{cases} \sin(\beta_{nr}x_r) - \dfrac{\sin(\beta_{nr}l)}{\sinh(\beta_{nr}l)}\sinh(\beta_{nr}x_r) & r = 1 \\ P_r M_r(x_r) + Q_r N_r(x_r) & r = 2,3,......N \end{cases} \tag{72}$$

$$P_r = \frac{[\cosh(\beta_{nr}l) - \cos(\beta_{nr}l)][\sin(\beta_{nr}l)\sinh(\beta_{nr}l)] - \sin(\beta_{nr}l)[\sinh(\beta_{nr}l)\cos(\beta_{nr}l) - \sin(\beta_{nr}l)\cosh(\beta_{nr}l)]}{\sinh(\beta_{nr}l)\left([\cos(\beta_{nr}l) - \cosh(\beta_{nr}l)][\sinh(\beta_{nr}l) - \sin(\beta_{nr}l)]\right)}$$

$$Q_r = \frac{[\cos(\beta_{nr}l) - \cosh(\beta_{nr}l)][\sin(\beta_{nr}l)\sinh(\beta_{nr}l)] + \sinh(\beta_{nr}l)[\sinh(\beta_{nr}l)\cos(\beta_{nr}l) - \sin(\beta_{nr}l)\cosh(\beta_{nr}l)]}{\sinh(\beta_{nr}l)\left([\cos(\beta_{nr}l) - \cosh(\beta_{nr}l)][\sinh(\beta_{nr}l) - \sin(\beta_{nr}l)]\right)}$$

$$M_r(x_r) = \left\{[\cos(\beta_{nr}l) - \cosh(\beta_{nr}l)]\sinh(\beta_{nr}x) + \sinh(\beta_{nr}l)[\cosh(\beta_{nr}x) - \cos(\beta_{nr}x)]\right\}$$

$$N_r(x_r) = \left\{[\cos(\beta_{nr}l) - \cosh(\beta_{nr}l)]\sin(\beta_{nr}x) + \sin(\beta_{nr}l)[\cosh(\beta_{nr}x) - \cos(\beta_{nr}x)]\right\}$$

.

The first five natural frequency parameters for two and three span beams are shown in Table 4. The length of each span is taken equal.

Table 4 First five frequency parameters for multispan beams

Number of Spans (r)	Frequency parameters (β_{nr})				
	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
1	π	2π	3π	4π	5π
2	3.1416	3.9272	6.2832	7.0686	9.4248
3	3.1416	3.5500	4.3040	6.2832	6.6920

The natural frequencies ω_{nr} can be found as $\omega_{nr} = (\beta_{nr} l)^2 \sqrt{\frac{EI}{ml^4}}$

The mode shapes of continuous beams with two and three equal spans are shown in Fig.8 and Fig.9

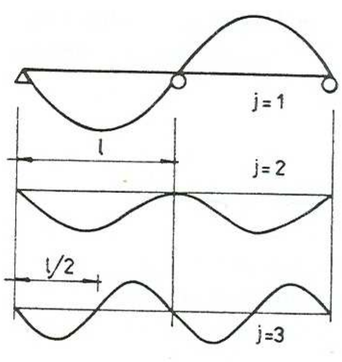


Fig.8 First two Mode shapes of continuous beam of two equal spans

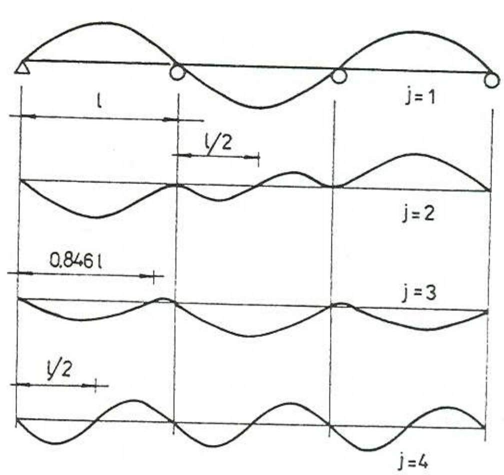


Fig.9 First three Mode shapes of continuous beam of two equal spans

APPROXIMATE METHODS

The approximate methods are necessary when the exact solution of differential equations can not be obtained such as in case of non uniform geometry, presence of concentrated masses and other non classical boundary conditions. Among them Rayeligh-Ritz method and Gallerkin method are the most popular in the study of continuous system. The success of these two methods depends on the choice of shape function that need to satisfy the geometrical boundary conditions. In this lecture, Rayeligh-Ritz method and Gallerkin methods will be decribed.

Rayeligh-Ritz method

Let U_{max} and T_{max} be the potential and kinetic energy of the system undergoing simple harmonic motion in free vibration.
Then

$$\omega^2 = \frac{U_{max}}{T_{max}} \tag{73}$$

Let the shape function be

$$w(x) = C_1\phi_1(x) + C_2\phi_2(x) + ... + C_n\phi_n(x) \tag{74}$$

where Ci are the constants and $\phi_I(x)$ are the admissible functions satisfying the boundary conditions. The maximum K.E and P. E are expressed as

$$U = \frac{1}{2}\sum_i\sum_j k_{ij}C_iC_j \text{ and } T = \frac{1}{2}\sum_i\sum_j m_{ij}C_iC_j \tag{75}$$

kij and mij depends on the type of problem. For example, for the beam we have

$$k_{ij} = \int EI\phi_i''\phi_j''\,dx \text{ and } m_{ij} = \int m\phi_i\phi_j\,dx \tag{76}$$

where as for longitudinal vibration of bars

$$k_{ij} = \int EA\phi_i''\phi_j''\,dx \text{ and } m_{ij} = \int m\phi_i\phi_j\,dx \tag{77}$$

We now minimize ω^2 by differentiating it with respect to each of the constants. Thus

$$\frac{\partial\omega^2}{\partial C_i} = \frac{T_{max}\dfrac{\partial U_{max}}{\partial C_i} - U_{max}\dfrac{\partial T_{max}}{\partial C_i}}{T_{max}{}^2} = 0 \tag{78}$$

which is satisfied by

$$\frac{\partial U_{max}}{\partial C_i} - \frac{U_{max}}{T_{max}}\frac{\partial T_{max}}{\partial C_i} = 0 \tag{79}$$

Using eq.(73), we have

$$\frac{\partial U_{max}}{\partial C_i} - \omega^2 \frac{\partial T_{max}}{\partial C_i} = 0 \tag{80}$$

The two terms of the equations are then

$$\frac{\partial U_{max}}{\partial C_i} = \sum_j k_{ij} C_j \text{ and } \frac{\partial T_{max}}{\partial C_i} = \sum_j m_{ij} C_j \tag{81}$$

and so finally we get

$$C_1(k_{i1} - \omega^2 m_{i1}) + C_2(k_{i2} - \omega^2 m_{i2}) + ... + C_n(k_{in} - \omega^2 m_{in}) = 0 \tag{82}$$

With i varying from 1 to n there will be n such equations, which can be arranged in matrix form

$$([K] - \omega^2[M])\{C\} = 0 \tag{83}$$

For non-trivial solution, the determinant of the matrix is equated to zero and characteristic polynomial can be obtained. The solution gives n natural frequencies and corresponding to each natural frequency, the {C} is found to obtain the mode shape function.

Gallerkin Method

In this method, we need to know the governing differential equations of the continuous system. Again, the shape function is chosen such that it satisfies the geometric boundary conditions.

Consider the eigen value problem

$$L[w] = \lambda M[w] \tag{84}$$

Here L and M are self adjoint homogeneous differential operators of order 2p and 2q. The function w can be taken in the form series of n comparison function as stated in the previous method. Substituting, the series of comparison functions in the differential equations, an error will be obtained.

$$\varepsilon = L[w_n] - \lambda^* M[w] \tag{85}$$

where λ^* is the corresponding estimate of the eigenvalue λ. Considering the orthogonality of the error with the assumed functions, Gallerkin's equation is obtained as

$$\int_D \varepsilon \phi_k dD = 0 \text{, k=1,2,..,n} \tag{86}$$

The following relations are obtained

$$\int_D \phi_i L[w_n] dD = \sum_{i=1}^{n} C_i \int_D L[\phi_j] dD = \sum_{j=1}^{n} K_{ij} C_j \tag{87}$$

and similarly $\int_D \phi_i M[w_n] dD == \sum_{j=1}^{n} m_{ij} C_j$ (88)

Which leads to

$$\sum_{j=1}^{n} (K_{ij} - \lambda^* m_{ij}) C_j = 0 \text{, I=1, 2,…,n} \quad (89)$$

Expansion of the eq.(89) results a similar matrix equation as in eq.(83), which can be solved as stated before.

CLOSURE

The basic dynamics of continuous system has been discussed. The inertia and stiffness and damping are distributed along the domain. The solution of free vibration problem leading to natural frequencies and mode shapes are outlined by exact method and by approximate method. The forced vibration problem is also discussed with the help of mode superposition of method. Torsional and Axial vibrations of bars (represented by second order partial differential equations) and bending vibrations of beam (represented by fourth order partial differential equations) are illustrated with some examples.

Further Reading

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